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Fundamental cycle of a periodic box–ball system and solvable lattice models

Jun Mada[1], **Makoto Idzumi**[2] **and Tetsuji Tokihiro**[1]

[1] Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Tokyo 153-8914, Japan
[2] Department of Mathematics, Faculty of Education, Shimane University, Matsue 690-8504, Japan

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Abstract
We investigate the fundamental cycle of a periodic box–ball system (PBBS) from a relation between the PBBS and a solvable lattice model. We show that the fundamental cycle of the PBBS is obtained from eigenvalues of the transfer matrix of the solvable lattice model.

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1. Introduction

A cellular automaton (CA) is a discrete dynamical system consisting of a regular array of cells [1]. Each cell can only have a finite number of states and is updated in discrete time steps. Although the updating rule is simple, CAs often exhibit very complicated time evolution patterns and they have been investigated as good models for natural and/or social phenomena. The box–ball systems (BBSs) are a well-studied class of filter-type CA which are expressed as discrete dynamical systems of balls in an infinite array of boxes [2, 3]. One reason for interest in them is the soliton-like solutions which they support. In fact, the BBSs can be obtained from integrable nonlinear equations through ultradiscretization [4, 5]. It can also be obtained from a solvable lattice model at zero temperature [6–8].

A periodic box–ball system (PBBS) is a BBS subject to a periodic boundary condition [9]. Being composed of a finite number of boxes and balls, it takes on a finite number of patterns. Therefore, the time evolution is necessarily periodic and the fundamental cycle, i.e. the shortest period of the periodic motion, exists for any given state. We can obtain an explicit formula expressing the fundamental cycle of a given state [10]. Using the formula, we can estimate the asymptotic behaviour of the fundamental cycles which shows an interesting number theoretical aspect of the PBBS [11, 12].

The PBBS is obtained through ultradiscretization of the nonautonomous discrete KP equation imposing the periodic boundary condition [13]. On the other hand, it can be obtained from the zero temperature limit of solvable lattice models which are generalizations of the six-vertex model; we noticed that the transfer matrix in this limit describes the time evolution of the PBBS and the fundamental cycle could be read off from the spectrum of the transfer matrix (announced in [12]). In this paper, we report results obtained from the investigations along this line.

The paper is organized as follows. In section 2, we give the definition of the PBBS. We describe conserved quantities which are expressed by a Young diagram, and give an explicit formula for the fundamental cycle which was obtained in [10]. In section 3, we explain why we can obtain the information on the fundamental cycle from the spectrum of the transfer matrix of the vertex models at zero temperature. In section 4, we give an explicit formula for the eigenvalue of the transfer matrix. To diagonalize the transfer matrix, we follow the standard framework of the Bethe ansatz method. The section ends with a conjecture concerning an essential relation between the conserved quantities of the PBBS and the solutions to the Bethe ansatz equation. Section 5 is devoted to conclusions and remarks.

This work is a continuation of our previous works on PBBSs [13, 14] and the main results were announced at the annual meeting of the Mathematical Society of Japan [21] and published in its collection of abstracts. After completing this work, one of the authors (TT) received a preprint treating the same system from a similar point of view [20], in which our theorem 3.1 together with corollary 4.1 were conjectured, and theorem 4.1 for general $A_M^{(1)}$ cases was stated. (See also remark 3.1 in ours.) In this paper we gave derivations of theorem 4.1 in more detail, which may help interested readers to understand the technical details.

2. Periodic box–ball system and the fundamental cycle

Consider a one-dimensional array of boxes each with a capacity of one ball. A periodic boundary condition is imposed by assuming that the last box is adjacent to the first one (e.g., the boxes are arranged in a circle). We denote the number of boxes by N and the number of balls by M. We assume $M \leqslant N/2$. An arrangement of M balls in N boxes is called a pattern or a state of the PBBS. The rule for the time evolution from time step t to $t+1$ is given as follows (see figure 1):

1. In each filled box, create a copy of the ball.
2. Move all the copies once according to the following rules.
3. Choose one of the copies and move it to the nearest empty box on the right of it.
4. Choose one of the remaining copies and move it to the nearest empty box on the right of it.
5. Repeat step 4 until all of the copies have moved.
6. Delete all the original balls.

An example of the time evolution of the PBBS according to this rule is shown in figure 2. Since there are a finite number of states and the time evolution rule is invertible, every trajectory is cyclic. The fundamental cycle of a given state is defined to be the length of the trajectory to which the state belongs.

An explicit formula is known for the fundamental cycle of a given state of the PBBS [10]. It is described in terms of conserved quantities. The algorithm to construct the conserved quantities is as follows [9, 15]. Denoting a vacant box by 0 and a filled box by 1, a state of the PBBS is represented as a 0,1 sequence of length N (see figure 3).

1. Let p_1 be the number of 10s in the sequence.
2. Eliminate all the 10s in the original sequence and let p_2 be the number of 10s in the new sequence.

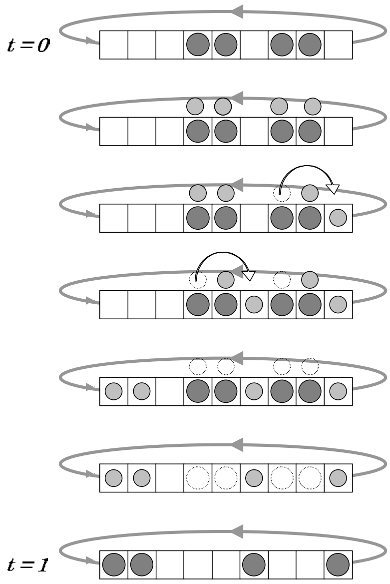

Figure 1. Time evolution rule for the PBBS.

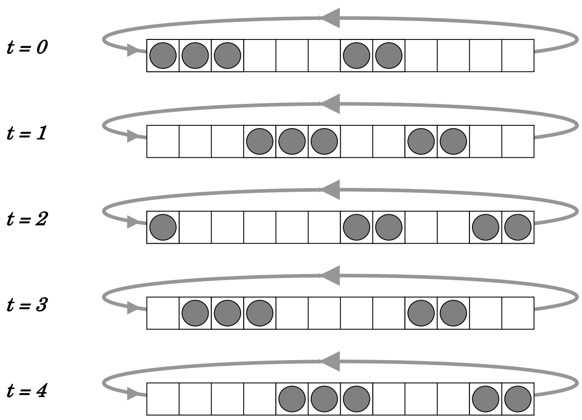

Figure 2. An example of the time evolution of the PBBS.

3. Similarly, let p_i denote the number of times the string 10 appears in the sequence at step i, where each step is followed by the deletion of all of the 10s which appear.
4. Let $\{p_1, p_2, \ldots, p_m\}$ be the weakly decreasing finite sequence of positive integers constructed in this way with $p_{m+1} = 0$ being the first occurrence of zero.

For example, for the state

$$(\sharp) \quad 0111111000111000110000100001000110000,$$

we have $p_1 = 6$, and eliminating 10s, we obtain a new sequence

$$0111100110010000000001000$$

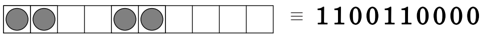

Figure 3. A correspondence between a state of the PBBS and a 0, 1 sequence.

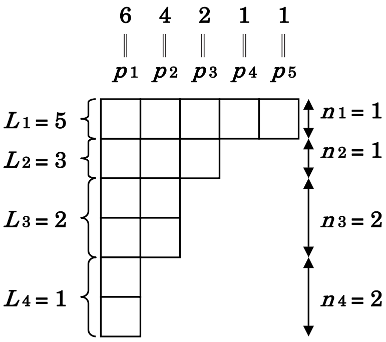

Figure 4. Young diagram corresponding to the conserved quantities of ($\sharp$).

and $p_2 = 4$. In a similar manner, we have $p_3 = 2$, $p_4 = 1$, $p_5 = 1$. To see that these $\{p_j\}$ are conserved, we evolve ($\sharp$) by one time step

$$(\sharp') \quad 000000111000111001111010000100001100.$$

By applying the above algorithm again, we find the same integer sequence $\{p_j\}_{j=1}^{5}$.

As the sequence $\{p_1, p_2, \ldots, p_m\}$ is weakly decreasing, we can associate a Young diagram with it by regarding p_j as the number of squares in the jth column of the diagram, see figure 4. The lengths of the rows are also weakly decreasing positive integers. Let the distinct row lengths be $L_1 > L_2 > \cdots > L_s$ and let n_j be the number of times that length L_j appears (see figure 4). The set $\{L_j, n_j\}_{j=1}^{s}$ is another expression for the conserved quantities of the PBBS.

Let $\ell_0 := N - 2M$, $N_0 := \ell_0$, $L_{s+1} := 0$ and

$$\ell_j := L_j - L_{j+1} \quad (j = 1, 2, \ldots, s),$$
$$N_j := \ell_0 + \sum_{k=1}^{j} 2n_k(L_k - L_{j+1}) \quad (j = 1, 2, \ldots, s).$$

Then, for the fundamental cycle, the following proposition holds:

Proposition 2.1 ([10]). *The fundamental cycle is a divisor of*

$$\mathrm{LCM}\left(\frac{N_s N_{s-1}}{\ell_s \ell_0}, \frac{N_{s-1} N_{s-2}}{\ell_{s-1} \ell_0}, \ldots, \frac{N_1 N_0}{\ell_1 \ell_0}, 1\right), \tag{2.1}$$

where $\mathrm{LCM}(x, y) := 2^{\max[x_2, y_2]} 3^{\max[x_3, y_3]} 5^{\max[x_5, y_5]} \cdots$ *for* $x = 2^{x_2} 3^{x_3} 5^{x_5} \cdots$ *and* $y = 2^{y_2} 3^{y_3} 5^{y_5} \cdots$ *with* $x_i, y_i \in \mathbb{Z}$*, and* $\mathrm{LCM}(x, y, z, \ldots) := x, \mathrm{LCM}(\mathrm{LCM}(y, z, \ldots))$.

Since (2.1) depends on the Young diagram Y expressing the conserved quantities of the PBBS, we denote the LCM by $T(Y)$.

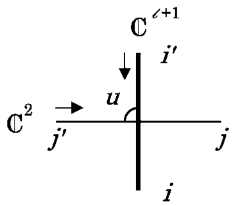

Figure 5. $\langle i, j|R^{(\ell,1)}(u;\lambda)|i', j'\rangle$.

3. Relation between the PBBS and solvable lattice models

In this section we explain that the time evolution of the PBBS is given by the transfer matrix of a vertex model.

Consider the six-vertex model and its higher-spin generalizations. A method of constructing the R-matrices $R^{(\ell,\ell')}$ is described in [16]; $R^{(\ell,\ell')}: \mathbb{C}^{\ell+1}\otimes\mathbb{C}^{\ell'+1}\to\mathbb{C}^{\ell+1}\otimes\mathbb{C}^{\ell'+1}$

$$R^{(\ell,\ell')}(|i'\rangle\otimes|j'\rangle) = \sum_{i=0}^{\ell}\sum_{j=0}^{\ell'}\langle i, j|R^{(\ell,\ell')}|i', j'\rangle(|i\rangle\otimes|j\rangle)$$

where $\{|i\rangle | i = 0, 1, \ldots, \ell\}$ is a basis of $\mathbb{C}^{\ell+1}$ and $\{|j\rangle | j = 0, 1, \ldots, \ell'\}$ that of $\mathbb{C}^{\ell'+1}$. The R-matrices are parametrized by two parameters, u and λ. For $\ell' = 1$, the parametrization of $R^{(\ell,1)} = R^{(\ell,1)}(u;\lambda)$ which we need is given by

(i) for $k = 0, 1, \ldots, \ell$,

$$\langle k, 1|R^{(\ell,1)}(u;\lambda)|k, 1\rangle = \rho\sinh\left(\lambda\left(k-\frac{\ell-1}{2}\right)+u\right),$$

$$\langle k, 0|R^{(\ell,1)}(u;\lambda)|k, 0\rangle = \rho\sinh\left(\lambda\left(\frac{\ell+1}{2}-k\right)+u\right),$$

(ii) for $k = 1, 2, \ldots, \ell$,

$$\langle k-1, 1|R^{(\ell,1)}(u;\lambda)|k, 0\rangle = \langle k, 0|R^{(\ell,1)}(u;\lambda)|k-1, 1\rangle = \rho\sqrt{\sinh\lambda k\sinh\lambda(\ell-k+1)},$$

(iii) otherwise, $\langle i, j|R^{(\ell,1)}(u;\lambda)|i', j'\rangle = 0$

where u and λ are called the spectral parameter and the deformation parameter, respectively. The normalization constant $\rho = \rho(u;\lambda)$ is chosen such that $\langle \ell, 1|R^{(\ell,1)}(u;\lambda)|\ell, 1\rangle = 1$. The matrix $R^{(1,\ell)}(u;\lambda)$ is related to $R^{(\ell,1)}(u;\lambda)$ by

$$R^{(1,\ell)}(u;\lambda) = P^{(\ell,1)}R^{(\ell,1)}(u;\lambda)P^{(1,\ell)},$$

where $P^{(m,n)}: \mathbb{C}^{m+1}\otimes\mathbb{C}^{n+1}\to\mathbb{C}^{n+1}\otimes\mathbb{C}^{m+1}$ is the permutation

$$P^{(m,n)}(|i\rangle\otimes|j\rangle) = |j\rangle\otimes|i\rangle.$$

These R-matrices satisfy the Yang–Baxter equation (YBE)

$$R_{12}^{(1,\ell)}(u;\lambda)R_{13}^{(1,1)}(u+v;\lambda)R_{23}^{(\ell,1)}(v;\lambda) = R_{23}^{(\ell,1)}(v;\lambda)R_{13}^{(1,1)}(u+v;\lambda)R_{12}^{(1,\ell)}(u;\lambda) \tag{3.1}$$

where $R_{12} = R\otimes I$, $R_{23} = I\otimes R$ and $R_{13} = \sum a\otimes 1\otimes b\big(R = \sum a\otimes b\big)$ (see figures 5 and 6).

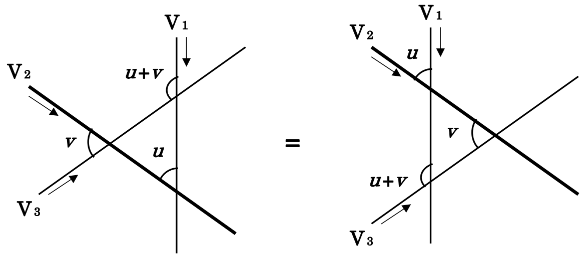

Figure 6. A graphical representation of the YBE (3.1). ($V_1 = \mathbb{C}^2$, $V_2 = \mathbb{C}^{\ell+1}$ and $V_3 = \mathbb{C}^2$.)

In the limit $\lambda \to +\infty$, the R-matrix given in [16] tends to a matrix which describes the local time evolution of a BBS. Let $\mathcal{R} := \lim_{u\to 0} \lim_{\lambda\to+\infty} R^{(1,\ell)}(u;\lambda)$. Then its action is given by

$$\mathcal{R}: \begin{cases} |1\rangle \otimes |\ell\rangle & & \mapsto & |1\rangle \otimes |\ell\rangle, \\ |1\rangle \otimes |k\rangle & (0 \leqslant k < \ell) & \mapsto & |0\rangle \otimes |k+1\rangle, \\ |0\rangle \otimes |k\rangle & (0 < k \leqslant \ell) & \mapsto & |1\rangle \otimes |k-1\rangle, \\ |0\rangle \otimes |0\rangle & & \mapsto & |0\rangle \otimes |0\rangle. \end{cases}$$

Hence, if we regard

- $|1\rangle, |0\rangle \in \mathbb{C}^2$ as the filled box and the vacant box, respectively, and
- $|k\rangle \in \mathbb{C}^{\ell+1} (k = 0, 1, \ldots, \ell)$ as the carrier with capacity ℓ carrying k balls,

the action of $\mathcal{R}$ coincides with the local time evolution rule of the BBS with a carrier of capacity ℓ, introduced in [17]. Hereafter, we assume that ℓ is larger than the number of balls. In this case the BBS with the carrier coincides with the original BBS.

Define the transfer matrix $\hat{t}(u;\lambda) : V \to V (V := \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{N})$ by

$$\begin{aligned} &\hat{t}(u;\lambda)(|i'_1\rangle \otimes |i'_2\rangle \otimes \cdots \otimes |i'_N\rangle) \\ &\qquad = \sum_{i_1,i_2,\ldots,i_N \in \{0,1\}} \langle i_1, i_2, \ldots, i_N | \hat{t}(u;\lambda) | i'_1, i'_2, \ldots, i'_N\rangle (|i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_N\rangle), \\ &\langle i_1, i_2, \ldots, i_N | \hat{t}(u;\lambda) | i'_1, i'_2, \ldots, i'_N\rangle \\ &\qquad := \sum_{j_1,j_2,\ldots,j_N=0}^{\ell} \langle i_N, j_1 | R^{(1,\ell)}(u;\lambda) | i'_N, j_N\rangle \langle i_{N-1}, j_N | R^{(1,\ell)}(u;\lambda) | i'_{N-1}, j_{N-1}\rangle \cdots \\ &\qquad\quad \cdots \langle i_2, j_3 | R^{(1,\ell)}(u;\lambda) | i'_2, j_2\rangle \langle i_1, j_2 | R^{(1,\ell)}(u;\lambda) | i'_1, j_1\rangle. \end{aligned} \tag{3.2}$$

We also define $V_{[M]} := \mathrm{span}\{|i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_N\rangle \in V \,|\, i_1 + i_2 + \cdots + i_N = M\}$. Since the R-matrix obeys the so-called ice condition

$$\langle i, j | R^{(1,\ell)} | i', j'\rangle = 0 \qquad \text{unless} \quad i + j = i' + j',$$

$\hat{t}(u;\lambda)$ maps $V_{[M]}$ into itself. Let $\hat{t} := \lim_{u\to 0} \lim_{\lambda\to+\infty} \hat{t}(u;\lambda)$, and let $\Omega_{[M]}$ be the set consisting of all 0, 1 sequences of length N such that the number of 1s is M. Note that $\Omega_{[M]}$ is identified with the set of states of the PBBS with M balls. We regard $\Omega_{[M]} \subset V_{[M]}$ by identifying a monomial $|i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_N\rangle \in V_{[M]}$ with a state $i_1 i_2 \cdots i_N \in \Omega_{[M]}$. Then, $\hat{t}|_{\Omega_{[M]}}$ maps $\Omega_{[M]}$ onto itself, and it gives the time evolution of the PBBS [9]. For example $00110010 \stackrel{\hat{t}}{\longmapsto} 00001101$.

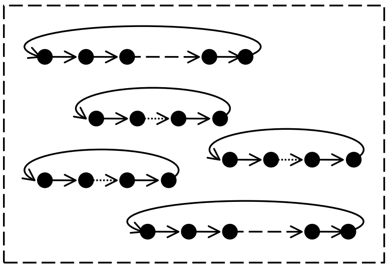

Figure 7. Decomposition of $\Omega_{[M]}$ into trajectories. ('●' and '→' represent the element of $\Omega_{[M]}$ and the action of $\hat{t}$, respectively.)

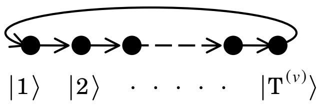

Figure 8. $\hat{t}$ is diagonalized on each trajectory as $\hat{t}|\tilde{k}\rangle = \mathrm{e}^{2\pi\sqrt{-1}k/T^{(\nu)}}|\tilde{k}\rangle$ where $|\tilde{k}\rangle = \sum_{n=1}^{T^{(\nu)}} \mathrm{e}^{-2\pi\sqrt{-1}kn/T^{(\nu)}}|n\rangle (k = 1, 2, \ldots, T^{(\nu)})$.

Next we will explain how knowledge of the spectrum of the transfer matrix allows one to determine the fundamental cycles. As was mentioned in the previous section, $\Omega_{[M]}$ is decomposed into the cyclic trajectories of $\hat{t}$

$$\Omega_{[M]} = \bigsqcup_{\nu} \Omega^{(\nu)},$$

where $\Omega^{(\nu)}$ denotes a set of states in a trajectory ν, and $T^{(\nu)} = |\Omega^{(\nu)}|$ is by definition the fundamental cycle of a state in $\Omega^{(\nu)}$ (figure 7). As is easily seen, the spectrum of $\hat{t}|_{\Omega^{(\nu)}}$ is $\{\exp(2\pi\sqrt{-1}k/T^{(\nu)})|k = 1, 2, \ldots, T^{(\nu)}\}$ (see figure 8). The connection between the spectrum of $\hat{t}\big|_{\Omega_{[M]}}$ and the fundamental cycles is given in the following theorem.

Theorem 3.1. *Let the eigenvalues of the restriction of $\hat{t}$ to the subspace spanned by the members in $\Omega^{(\nu)}$ be written as*

$$\exp\left(2\pi\sqrt{-1}\frac{Q_k}{P_k}\right) \qquad (k = 1, 2, \ldots, T^{(\nu)})$$

where Q_k and P_k are coprime for each k. Then each P_k is a divisor of the fundamental cycle $T^{(\nu)}$, and

$$\max\{P_k|k = 1, 2, \ldots, T^{(\nu)}\} = T^{(\nu)}.$$

Remark 3.1. Theorem 3.1 is a consequence of the following general fact; when a transfer matrix gives an updating rule of an *invertible* discrete dynamical system, its eigenvalues are given as

$$\exp\left[2\sqrt{-1}\pi\frac{k}{T_\nu}\right] \qquad (k = 0, 1, 2, \ldots, T_\nu - 1),$$

where T_ν is a fundamental cycle for an orbit indexed by a parameter ν. Although this fact is simple and easily understood, it gives a universal relationship between fundamental cycles of a cellular automaton and the lattice model associated with it.

4. Diagonalization of the transfer matrix

In the following, we use

$$x = \mathrm{e}^{u}, \qquad q = \mathrm{e}^{-\lambda}$$

instead of u and λ. Accordingly, we write $R[x;q] = R(u;\lambda)$ and so on.

The transfer matrix (3.2) commutes with the transfer matrix of the six-vertex model by virtue of the YBE (3.1). It follows that they have a common set of eigenvectors. Let $|\varphi\rangle$ be a vector in $V(=\underbrace{\mathbb{C}^2\otimes\mathbb{C}^2\otimes\cdots\otimes\mathbb{C}^2}_{N})$ of the form

$$|\varphi\rangle = \left|\varphi;\{x_j\}_{j=1}^{M}\right\rangle = C[x_1;q]C[x_2;q]\cdots C[x_M;q](|0\rangle\otimes|0\rangle\otimes\cdots\otimes|0\rangle),$$

where $C[x;q] : V \to V$ is a creation operator in the algebraic Bethe ansatz method [18], which is defined by

$$\begin{aligned}C[x;q](|i'_1\rangle\otimes|i'_2\rangle\otimes\cdots\otimes|i'_N\rangle)&\\ = \sum_{i_1,i_2,\ldots,i_N\in\{0,1\}}&\langle i_1,i_2,\ldots,i_N|C[x;q]|i'_1,i'_2,\ldots,i'_N\rangle(|i_1\rangle\otimes|i_2\rangle\otimes\cdots\otimes|i_N\rangle),\end{aligned}$$

$$\begin{aligned}\langle i_1,i_2,\ldots,i_N|C[x;q]|i'_1,i'_2,\ldots,i'_N\rangle&\\ := \sum_{j_2,j_3,\ldots,j_N=0}^{1}&\langle i_N,1|R^{(1,1)}[x;q]|i'_N,j_N\rangle\langle i_{N-1},j_N|R^{(1,1)}[x;q]|i'_{N-1},j_{N-1}\rangle\cdots\\ &\cdots\langle i_2,j_3|R^{(1,1)}[x;q]|i'_2,j_2\rangle\langle i_1,j_2|R^{(1,1)}[x;q]|i'_1,0\rangle.\end{aligned}$$

Assume that the additional parameters $x_1, x_2, \ldots, x_M$ are mutually distinct and satisfy the Bethe ansatz equation (BAE)

$$\left(\frac{q^{-1}x_k - qx_k^{-1}}{x_k - x_k^{-1}}\right)^{N} = \prod_{\substack{j=1\\ j\neq k}}^{M}\frac{q^{-1}x_k x_j^{-1} - qx_k^{-1}x_j}{qx_kx_j^{-1} - q^{-1}x_k^{-1}x_j}. \tag{4.1}$$

Then $\hat{t}[x;q]|\varphi\rangle = \Lambda[x;\{x_j\};q]|\varphi\rangle$, where

$$\begin{aligned}\Lambda[x;\{x_j\};q] = \sum_{k=0}^{\ell}&\left(\frac{q^kx - q^{\ell-k+1}x^{-1}}{x - q^{\ell+1}x^{-1}}\right)^{N}\\ &\times\prod_{j=1}^{M}\frac{\left(q^{-1}x_j^2\right)x^{-2} - q^{-\ell-2} - q^{\ell} + q^{-2}\left(q^{-1}x_j^2\right)^{-1}x^2}{q^{\ell-2k}\left(q^{-1}x_j^2\right)x^{-2} - 1 - q^{-2} + q^{2k-\ell-2}\left(q^{-1}x_j^2\right)^{-1}x^2}.\end{aligned}$$

For any solution $\{x_k\}_{k=1}^{M}$ of (4.1), each x_k is a function of q. In relation to the PBBS, we are interested in these solutions at $q = 0$. To do this, we will make use of the string hypothesis. First, let Y be the Young diagram, which represents the partition of M

$$\underbrace{m_1 + m_1 + \cdots + m_1}_{K_1} + \underbrace{m_2 + m_2 + \cdots + m_2}_{K_2} + \cdots + \underbrace{m_s + m_s + \cdots + m_s}_{K_s} \tag{4.2}$$

where $m_1 < m_2 < \cdots < m_s$ and $K_i > 0\,(i = 1,2,\ldots,s)$ (figure 9). Then, the string hypothesis is the assumption that any solution $\{x_j\}_{j=1}^{M}$ to the BAE (4.1) is expressed as $\{x_{i\alpha k}\}$ of the form

$$(x_{i\alpha k}(q))^2 = q^{m_i-2k+2}\{z_{i\alpha}^0 + O(q)\} \qquad (i = 1,2,\ldots,s;\ \alpha = 1,2,\ldots,K_i;\ k = 1,2,\ldots,m_i).$$

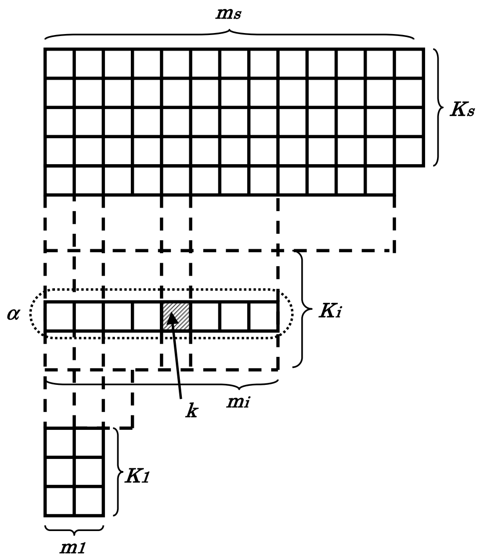

Figure 9. The Young diagram corresponding to (4.2).

Accordingly, the space $V_{[M]}$ decomposes into subspaces $V_{[M]}^{Y}(q)$ spanned by the eigenvectors determined by the string hypothesis:

$$V_{[M]} = \bigoplus_{Y} V_{[M]}^{Y}(q),$$

$$V_{[M]}^{Y}(q) = \mathrm{span}\{|\varphi; \{x_{i\alpha k}\}\rangle \mid \{x_{i\alpha k}\} \in S_Y\}$$

where S_Y is the set of all solutions to the BAE for a given Young diagram Y. By direct calculation we obtain the following expression.

Proposition 4.1. *Eigenvalues of the transfer matrix* $\hat{t} = \lim_{x\to 1}\lim_{q\to 0}\hat{t}[x;q]$ *corresponding to the eigenvectors in* $V_{[M]}^{Y} := \lim_{q\to 0} V_{[M]}^{Y}(q)$ *are given by*

$$\prod_{i=1}^{s}\prod_{\alpha=1}^{K_i}\left(-z_{i\alpha}^{0}\right)^{m_i}. \tag{4.3}$$

Hence, we need only the leading coefficients $\left\{z_{i\alpha}^{0}\right\}$ of the solution to the BAE. A system of equations which determines $\left\{z_{i\alpha}^{0}\right\}$ may be derived [19] from the BAE (4.1). This is called the string centre equation (SCE):

$$\prod_{j=1}^{s}\prod_{\beta=1}^{K_j}\left(z_{j\beta}^{0}\right)^{A_{i\alpha,j\beta}} = (-1)^{N+K_i+1} \qquad (i = 1,2,\ldots,s;\ \alpha = 1,2,\ldots,K_i) \tag{4.4}$$

where

$$A_{i\alpha,j\beta} := \delta_{ij}\delta_{\alpha\beta}(P_i + K_i) + 2\min(m_i, m_j) - \delta_{ij}, \qquad P_i := N - 2\sum_{j=1}^{s} K_j \min(m_i, m_j).$$

In terms of the real variables $u_{i,\alpha}$ defined by $z_{i,\alpha}^{0} = \exp(2\pi\sqrt{-1}u_{i,\alpha})$, the SCE (4.4) becomes a system of linear congruence equations. It is solved explicitly using Cramer's rule by introducing a set of integers $h = \{h_{i\alpha} \mid i = 1,2,\ldots,s;\ \alpha = 1,2,\ldots,K_i\}$.

Theorem 4.1. *The eigenvalue (4.3) is*

$$\Lambda(Y;h)=\exp\left[\pi\sqrt{-1}\left(\frac{\sum_{i=1}^{s}\sum_{\alpha=1}^{K_i}m_i\det(B_{i\alpha})}{\det(A)}+M\right)\right]$$

where

$$A=\begin{bmatrix}A_{11,11}&\cdots&A_{11,i\alpha}&\cdots&A_{11,sK_s}\\A_{12,11}&\cdots&A_{12,i\alpha}&\cdots&A_{12,sK_s}\\\vdots&\ddots&\vdots&\ddots&\vdots\\A_{sK_s,11}&\cdots&A_{sK_s,i\alpha}&\cdots&A_{sK_s,sK_s}\end{bmatrix},$$

$$B_{i\alpha}=B_{i\alpha}(h)=\begin{bmatrix}&&\overset{(i,\alpha)}{\vee}&&\\A_{11,11}&\cdots&N+K_1+2h_{11}+1&\cdots&A_{11,sK_s}\\A_{12,11}&\cdots&N+K_1+2h_{12}+1&\cdots&A_{12,sK_s}\\\vdots&\ddots&\vdots&\ddots&\vdots\\A_{sK_s,11}&\cdots&N+K_s+2h_{sK_s}+1&\cdots&A_{sK_s,sK_s}\end{bmatrix}.$$

Corollary 4.1. *It holds that*

$$(\Lambda(Y;h))^{T(Y)}=1.$$

Proof. One can obtain immediately

$$\det(A)=\left(N\prod_{j=1}^{s-1}P_{m_j}\right)\prod_{i=1}^{s}\left(P_{m_i}+K_{m_i}\right)^{K_{m_i}-1},$$

and

$$\sum_{i=1}^{s}\sum_{\alpha=1}^{K_i}m_i\det\left(B_{m_i\alpha}\right)=\det(\widetilde{B})\prod_{i=1}^{s}\left(P_{m_i}+K_{m_i}\right)^{K_{m_i}-1}-\det(A)$$

where

$$\widetilde{B}=\begin{bmatrix}P_1+m_1(H_1+2)&2m_1+m_2H_1&2m_1+m_3H_1&\cdots&2m_1+m_sH_1\\m_1(H_2+2)&P_2+m_2(H_2+2)&2m_2+m_3H_2&\cdots&2m_2+m_sH_2\\m_1(H_3+2)&m_2(H_3+2)&P_3+m_3(H_3+2)&\cdots&2m_3+m_sH_3\\m_1(H_4+2)&m_2(H_4+2)&m_3(H_4+2)&\cdots&2m_4+m_sH_4\\\vdots&\vdots&\vdots&\vdots&\vdots\\m_1(H_s+2)&m_2(H_s+2)&m_3(H_s+2)&\cdots&P_s+m_s(H_s+2)\end{bmatrix}$$

and $H_j=\sum_{\alpha=1}^{K_j}(N+K_j+2h_{j\alpha}+1)$. The determinant $\det(\widetilde{B})$ is calculated as

$$\det(\widetilde{B})-N\prod_{j=1}^{s-1}P_{m_j}=\left(m_1\prod_{\ell=2}^{s}P_\ell\right)\sum_{i=1}^{s}H_i+\left(N\prod_{\ell=1}^{s}P_\ell\right)\sum_{j=2}^{s}\frac{(m_j-m_{j-1})\sum_{i=j}^{s}H_i}{P_{j-1}P_j}. \tag{4.5}$$

In the derivation of (4.5), we have used several identities such as

$$P_j-\frac{m_j}{m_{j+1}}P_{j+1}=\left(N-2\sum_{i=1}^{j}m_iK_i\right)\left(1-\frac{m_j}{m_{j+1}}\right)\qquad(j=1,2,\ldots,s-1),$$

$$\left(N-2\sum_{i=1}^{k}m_iK_i\right)\left(1-\frac{m_j}{m_{j+1}}\right)+\frac{m_km_{k+1}(m_{j+1}-m_j)}{m_{k+1}(m_{k+1}-m_k)m_{j+1}}P_{k+1}=\frac{m_{k+1}(m_{j+1}-m_j)}{(m_{k+1}-m_k)m_{j+1}}P_k$$

$$(k=1,2,\ldots,s-2;\ j=k+1,k+2,\ldots,s-1)$$

which are obtained from the definition $P_i = N - 2\sum_{j=1}^{s} K_j \min(m_i, m_j)$. Finally, we obtain an expression

$$\Lambda(Y; h) = \exp[\pi\sqrt{-1}\Phi(Y; h)]$$

where

$$\Phi(Y; h) = \frac{m_1 P_s \sum_{i=1}^{s} H_i}{N P_1} + \sum_{j=2}^{s} \frac{(m_j - m_{j-1}) P_s \sum_{i=j}^{s} H_i}{P_{j-1} P_j} + M.$$

Now $T(Y)$, the LCM in proposition 2.1 for the Young diagram Y, can be expressed in terms of m_i and P_i:

$$T(Y) = \mathrm{LCM}\left(\frac{N P_1}{m_1 P_s}, \frac{P_1 P_2}{(m_2 - m_1) P_s}, \ldots, \frac{P_{s-1} P_s}{(m_s - m_{s-1}) P_s}, 1\right).$$

It is easily seen that $\Phi(Y; h) \times T(Y)$ is an integer. Let us show that $\Phi(Y; h) \times T(Y)$ is even.

If N is odd, then P_j s are odd and $H_j \equiv K_j \bmod 2$; hence, the numerator of

$$\Phi(Y; h) = \frac{1}{N \prod_{j=1}^{s-1} P_j} \left\{ m_1 P_2 P_3 \cdots P_{s-1} P_s \sum_{i=1}^{s} H_i + \left(N \prod_{\ell=1}^{s} P_\ell\right) \sum_{j=2}^{s} \frac{(m_j - m_{j-1}) \sum_{i=j}^{s} H_i}{P_{j-1} P_j} + MN \prod_{j=1}^{s-1} P_j \right\}$$

is even and the denominator is odd; thus, $\Phi(Y; h) \times T(Y)$ is even. On the other hand, if N is even, then H_j are even and

$$\left(\frac{m_1 P_s}{N P_1} + \frac{(m_2 - m_1) P_s}{P_1 P_2} + \cdots + \frac{(m_s - m_{s-1})}{P_{s-1}}\right) \times T(Y)$$

is an integer; this means that $\{\Phi(Y; h) - M\} \times T(Y)$ is even; thus, it follows that if M is even or $T(Y)$ is even then $\Phi(Y; h) \times T(Y)$ is even. However, from the proof of the formula (2.1) given in [10], we can easily see that $T(Y)$ is always even when N is even. □

Proposition 2.1 and corollary 4.1 suggest the following conjecture. Let M be an integer and let Y be a Young diagram which gives a partition of M. Recall that $V_{[M]}^{Y}$ is a subspace of $V = \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{N}$ spanned by the eigenvectors of $\hat{t}$ corresponding to the Young diagram Y. On the other hand, $\Omega_{[M]}^{Y}$ is the set of states of the PBBS, whose member consists of M balls and has the conserved quantity characterized by Y described in section 2. Also recall that each element of $\Omega_{[M]}^{Y}$ is regarded as a monomial in V.

Conjecture. *If the string hypothesis is true, then*

$$V_{[M]}^{Y} \subset \mathrm{span}\, \Omega_{[M]}^{Y}.$$

Furthermore, if all the eigenvectors can be obtained by the string hypothesis, the stronger condition

$$V_{[M]}^{Y} = \mathrm{span}\, \Omega_{[M]}^{Y}$$

holds.

5. Concluding remarks

In this paper, we considered the fundamental cycle of the PBBS from the viewpoint of the relation between the PBBS and solvable lattice models, namely the six-vertex model and its generalizations to higher spin representations. Observing that in the zero temperature limit the transfer matrix of the lattice model gives exactly the time evolution of the PBBS, we showed that the fundamental cycles could be read from the spectrum of the transfer matrix. Using the Bethe ansatz method, we obtained a determinant formula for the eigenvalue at zero temperature, which should have information on the fundamental cycles. We conjectured that the Young diagram appearing in the string hypothesis exactly corresponds to the Young diagram characterizing the conserved quantities of the PBBS. We have a strong feeling on the validity of the conjecture.

We have considered the simplest PBBS, having only one species of ball, and box capacities being all equal to 1. Extension of our method to the PBBS with arbitrary box capacities (which could differ site by site) is straightforward: it could be done by using $R^{(\ell_k,\ell)}$ in (3.2), instead of $R^{(1,\ell)}$. Extension to the PBBS with many species of ball is also possible: in this case we have only to consider $U_q(sl_n)$ symmetry, instead of $U_q(sl_2)$.

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